Lit-only sigma-game and its dual game[*]

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Abstract

The lit-only σ-game is played on a simple graph $\Gamma = (S, R)$ of finite order n. Every vertex of Γ is assigned to one of the two colors, black or white, to form a configuration. A move on a configuration is to select a black vertex s of Γ and then to change those colors of all neighbors of s. It is known that on each simply-laced Dynkin diagram, for any initial configuration of the lit-only σ-game on Γ with at least one black vertex, one can find a sequence of moves to reach a configuration with a single black vertex. This leads to the combinatorial question: which graphs have the above property. This paper firstly gives a class of graphs, nondegenerate trees excluding types A_n for even n, satisfying that property other than the simply-laced Dynkin diagrams, and shows that the number of the orbits of the lit-only σ-game on such a graph is three. The proofs use the connection between the lit-only σ-game and another combinatorial game.

Keywords: combinatorial games; Coxeter systems; representations.

1 Introduction

Throughout this paper, let $\Gamma = (S, R)$ denote a finite simple graph of order n. Every vertex of Γ is assigned to one of the two colors, black or white, to form a *configuration*. A *move* on a configuration is to select a black vertex s of Γ and then to change those colors of all neighbors of s. This is the *lit-only σ-game* on Γ. Define a binary relation $\sim$ on the set of all configurations

on Γ by $f \sim g$ if and only if f can reach g by a sequence of moves. The binary relation $\sim$ is an equivalence relation and those equivalence classes are called the *orbits* of the lit-only σ-game on Γ. This game implicitly appears in the classification of Vogan diagrams. A *Vogan diagram* is a Dynkin diagram with two additional data: a diagram automorphism θ with $\theta^2 = 1$, and the vertices fixed by θ may be assigned black or white color. Each Vogan diagram corresponds to a real simple Lie algebra, and two Vogan diagrams are said to be equivalent if they represent the same Lie algebra. See [13] for details.

According to the Borel-de Siebenthal theorem [1], every Vogan diagram with at least one black vertex is equivalent to one with a single black vertex. In 2004, M. Chuah and C. Hu explicitly described the equivalence classes of Vogan diagrams in combinatorial ways [2]. Gerald Jennhwa Chang introduced this game to combinatorists by the talk "Graph Painting and Lie Algebra" in 2005 International and Third Cross-strait Conference on Graph Theory and Combinatorics. Subsequently Xinmao Wang and Yaokun Wu recognized this game as a variation of another game, called σ-game, appeared as early as in the 2001 paper [4] of H. Eriksson, K. Eriksson and J. Sjöstrand. The configurations of the σ-game on Γ are identical to those of the lit-only σ-game on Γ. A move of the σ-game on Γ is to select an arbitrary vertex s of Γ and then to change those colors of all neighbors of s. The following briefly introduces the difference between σ-games and lit-only σ-games below from the view of algebra.

Let V denote the n-dimensional vector space over the two-element field $\mathbf{F}_2$ with basis $\{\alpha_s \mid s \in S\}$. Every vector $\lambda = \sum_{s \in S} c_s \alpha_s$ of V can be regarded as the configuration consisting of all black vertices $s \in S$ with $c_s = 1$. In terms of linear algebra, for two configurations $\lambda = \sum_{s \in S} c_s \alpha_s$ and $\mu = \sum_{s \in S} d_s \alpha_s$, the configuration λ can reach μ by a sequence of moves of the σ-game on Γ if and only if the system of linear equations

$$A(x_s)_{s \in S} = (d_s - c_s)_{s \in S}$$

over $\mathbf{F}_2$ has a solution in the n variables x_s for $s \in S$, where A is the adjacency matrix of Γ over $\mathbf{F}_2$. It can be shown that the number of the orbits of the σ-game on Γ is equal to the size of the null space of A, and the size of each orbit is a constant equal to the size of the column space of A. See [6, 12, 15] for more information. However, from the view of algebra, lit-only σ-games are more complicated than σ-games.

Throughout this paper, let (W, S) denote the *Coxeter system* of type Γ. This means that W is the group with generating set S subject only to

relations of the form
$$(ss')^{m(s,s')} = 1$$
for $s, s' \in S$, where
$$m(s, s') = \begin{cases} 1, & \text{if } s = s'; \\ 3, & \text{if } ss' \in R; \\ 2, & \text{else.} \end{cases}$$

One of the most attractive features of lit-only σ-games is that there is a representation ϕ of W into the general linear group of degree n over $\mathbf{F}_2$ such that the natural action of $\phi(W)$ on the n-dimensional vector space corresponds to the lit-only σ-game on Γ. The number of the orbits of the lit-only σ-game on Γ and the size of each orbit are difficult to determine, even if the rank of the adjacency matrix A of Γ over $\mathbf{F}_2$ is known. See [10] for details. In addition, the 2005 paper [14] of M. Reeder gives a representation φ of W into the general linear group of degree n over $\mathbf{F}_2$, and the natural action of $\varphi(W)$ on the n-dimensional vector space corresponds to another combinatorial game called Reeder's game in this paper. Actually, the above two representations are dual representations and so Reeder's game can be viewed as the dual game of the lit-only σ-game. This paper shows all details in Sections 2 and 4.

The motivation of this paper comes from the combinatorial interpretation of the Borel-de Siebenthal theorem, which implies that each orbit of the lit-only σ-game on any simply-laced Dynkin diagram contains a configuration with at most one black vertex. This leads to the combinatorial question: which graphs have the above property. The main purpose of this paper is to show that a nondegenerate tree not of type A_n for some even n satisfies that property, and the number of the orbits of the lit-only σ-game on such a graph is three. The proofs principally use the knowledge of dual representations and the result of [14] on the representation φ in the case of nondegenerate trees excluding types A_n for even n. See Section 5 for details.

According to the main result of [16], if Γ is a tree with ℓ leaves, then each orbit of the lit-only σ-game on Γ contains a configuration with at most $\lceil \frac{\ell}{2} \rceil$ black vertices. Moreover, [16] gives examples to show the bound is sharp. As a result, the contribution of this paper also improves the above result in the case of nondegenerate trees excluding types A_n for even n. Note that a nondegenerate tree can be characterized by the graph-theoretic condition as a tree with a perfect matching. See Section 3 for details. Finally, Section 6 provides some open problems and reviews the progress in lit-only σ-games.

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Figure 1 shows the simply-laced Dynkin diagrams as follows.

$A_n(n \geq 1)$
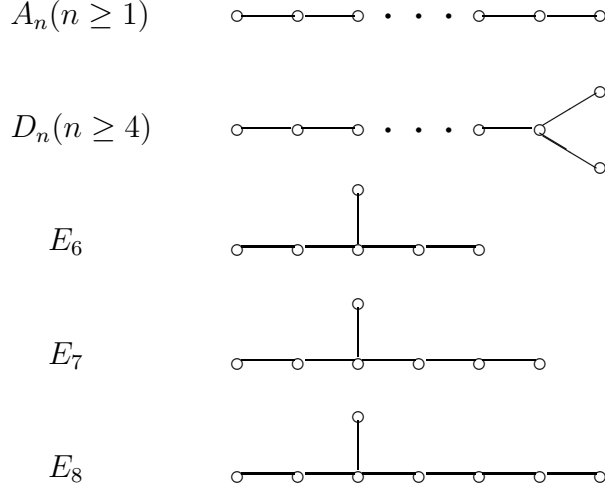
$D_n(n \geq 4)$

E_6

E_7

E_8

Figure 1: simply-laced Dynkin diagrams.

2   Reeder's game

This section introduces a few facts about Reeder's games appeared in the 2005 paper [14] of M. Reeder. The configurations of the Reeder's game on $\Gamma = (S, R)$ are identical to those of the lit-only σ-game on Γ. However, for Reeder's game, a move on a configuration is to select a vertex s of Γ with an odd number of black neighbors and then to change the color of s itself. Recall that V denotes the n-dimensional space over $\mathbf{F}_2$ with basis $\{\alpha_s \mid s \in S\}$. Let B be the symmetric bilinear form on V defined by

$$B(\alpha_s, \alpha_{s'}) = \begin{cases} 1, & \text{if } ss' \in R; \\ 0, & \text{else} \end{cases} \tag{2.1}$$

for $s, s' \in S$. Every vector $\lambda = \sum_{s \in S} c_s \alpha_s$ of V is regarded as the configuration of the Reeder's game on Γ consisting of all black vertices $s \in S$ with $c_s = 1$. An alternative interpretation for the moves in the Reeder's game on Γ is given as follows: on each round, select a vertex s of Γ. If s has an odd number of black neighbors, then change the color of s; otherwise, there is nothing to do. For each $s \in S$, observe that the new move selecting the vertex s corresponds

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to the linear transformation $\varphi_s : V \to V$ defined by

$$\varphi_s \lambda := \lambda - B(\alpha_s, \lambda)\alpha_s \qquad (2.2)$$

for $\lambda \in V$. It is clear that $\varphi_s^2 = 1$ and hence φ_s is in the general linear group $\mathrm{GL}(V)$ of V for $s \in S$. A direct calculation shows that the form B is invariant under φ_s for $s \in S$; i.e., $B(\varphi_s \lambda, \varphi_s \mu) = B(\lambda, \mu)$ for $\lambda, \mu \in V$.

The following theorem establishes a connection between the Coxeter system (W, S) of type Γ and the Reeder's game on Γ.

Theorem 2.1. *Suppose that (W, S) is the Coxeter system of type Γ. Then there is a unique homomorphism φ of W into $\mathrm{GL}(V)$ sending s to φ_s for all $s \in S$.*

Proof. Pick two distinct $s, s' \in S$. By (2.2), it is routine to check that

$$\begin{cases} (\varphi_s \varphi_{s'})^3 \alpha_t = \alpha_t, & \text{if } ss' \in R; \\ (\varphi_s \varphi_{s'})^2 \alpha_t = \alpha_t, & \text{else} \end{cases}$$

for $t \in S$. This shows $(\varphi_s \varphi_{s'})^3 = 1$ if $ss' \in R$ and $(\varphi_s \varphi_{s'})^2 = 1$ if $ss' \notin R$. Thus the assertion follows from the universal property. $\square$

In this paper, the homomorphism φ is called the *combinatorial representation of W of the first kind.* Recall that V is a W-module by defining

$$w\lambda := \varphi(w)\lambda$$

for $w \in W$ and $\lambda \in V$. Note that this paper shall use the language of modules along with the equivalent language of representations.

3   Combinatorial properties of a nondegenerate tree

Recall that the *radical* of the symmetric bilinear form B is the subspace

$$V^\perp := \{\lambda \in V \mid B(\lambda, \mu) = 0 \text{ for all } \mu \in V\}$$

of V. The form B is said to be *nondegenerate* if $V^\perp = \{0\}$. Otherwise, the form B is said to be *degenerate.* The following example determines whether the associated bilinear forms are nondegenerate in the case of types E_6 and E_7.

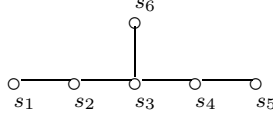**Example 3.1.** (i) Let Γ be the Dynkin diagram of type E_6 as follows.

Figure 2: the Dynkin diagram of type E_6.

Suppose that $\lambda = \sum_{i=1}^{6} c_i \alpha_{s_i}$ lies in the radical of the associated bilinear form B, where $c_i \in \mathbf{F}_2$ for $i = 1, 2, \ldots, 6$. From $B(\lambda, \alpha_{s_1}) = c_2$, $B(\lambda, \alpha_{s_6}) = c_3$ and $B(\lambda, \alpha_{s_5}) = c_4$, we determine $c_2 = c_3 = c_4 = 0$; i.e., $\lambda = c_1 \alpha_{s_1} + c_5 \alpha_{s_5} + c_6 \alpha_{s_6}$. From $B(\lambda, \alpha_{s_2}) = c_1$, $B(\lambda, \alpha_{s_4}) = c_5$ and $B(\lambda, \alpha_{s_3}) = c_6$, we further determine $c_1 = c_5 = c_6 = 0$ and hence $\lambda = 0$. Thus B is nondegenerate.

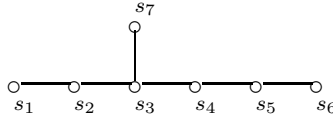(ii) Let Γ be the Dynkin diagram of type E_7 as follows.

Figure 3: the Dynkin diagram of type E_7.

It is straightforward to check that the vector $\alpha_{s_4} + \alpha_{s_6} + \alpha_{s_7}$ lies in the radical of the associated bilinear form B. Thus B is degenerate.

The graph Γ is said to be *nondegenerate* if its associated bilinear form B is nondegenerate. Otherwise, the graph Γ is said to be *degenerate*. Recall that Γ is said to be a *tree* if Γ is connected and has no cycles. A *leaf* of a tree is a vertex of degree 1. The following proposition provides an iterative criterion for nondegenerate trees. Note that, for $S' \subseteq S$, let $\Gamma - S'$ denote the subgraph of Γ obtained by deleting all vertices s in S' and their adjacent edges.

Proposition 3.2. *Suppose that $\Gamma = (S, R)$ is a tree of order at least two. Let $tt' \in R$ with t a leaf. Then the following conditions (i),(ii) are equivalent.*

(i) Γ is a nondegenerate tree.

(ii) Each connected component of $\Gamma - \{t, t'\}$ is a nondegenerate tree.

Proof. Let $\Gamma' = (S', R')$ denote a connected component of $\Gamma - \{t, t'\}$. Observe that the associated bilinear form B' of Γ' is equivalent to B restricted

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to the subspace V' of V spanned by $\{\alpha_s \mid s \in S'\}$, via the natural linear isomorphism. For convenience, the restriction of B to V' is regarded as B' in the following.

(i) $\Rightarrow$ (ii). Pick any connected component $\Gamma' = (S', R')$ of $\Gamma - \{t, t'\}$. Suppose that there is a nonzero vector $\mu = \sum_{s \in S'} c_s \alpha_s$ in the radical of the associated bilinear form of Γ'. Note that $B(\mu, \alpha_s) = 0$ for $s \in S - \{t'\}$, since every vertex in $S - S'$ is not adjacent to any vertex of Γ' except the vertex t' and by the definition (2.1) of B. Since μ is nonzero and by hypothesis (i), this forces $B(\mu, \alpha_{t'}) = 1$. But there is another nonzero vector $\mu + \alpha_t$ such that $B(\mu + \alpha_t, \alpha_s) = 0$ for all $s \in S$, which is a contradiction. Thus Γ' is nondegenerate.

(ii) $\Rightarrow$ (i). Suppose that $\lambda = \sum_{s \in S} c_s \alpha_s$ lies in the radical $V^\perp$ of the associated bilinear form B of Γ. Note that $c_{t'} = 0$, since $B(\lambda, \alpha_t) = c_{t'}$. Hence, for each connected component $\Gamma' = (S', R')$ of $\Gamma - \{t, t'\}$, the vector $\sum_{s \in S'} c_s \alpha_s$ lies in the radical of the associated bilinear form of Γ'. From this and the hypothesis (ii), we have $c_s = 0$ for all $s \in S - \{t\}$ and hence $\lambda = 0$ or α_t. Since $B(\alpha_t, \alpha_{t'}) = 1$, this forces $\lambda = 0$. Thus Γ is nondegenerate. $\square$

Example 3.1 follows definitions to determine the Dynkin diagram of type E_6 is nondegenerate, and the Dynkin diagram of type E_7 is not. The following example uses Proposition 3.2 to determine whether the Dynkin diagrams of types A_n, D_n and E_8 are nondegenerate.

Example 3.3. (i) It is straightforward to verify that the Dynkin diagram of type A_1 is degenerate and the Dynkin diagram of type A_2 is nondegenerate. Suppose that Γ is of type A_n as follows.

Figure 5: the Dynkin diagram of type A_n.

Note that the connected graph $\Gamma - \{s_{n-1}, s_n\}$ is of type A_{n-2}. Hence, by an induction on n and Proposition 3.2, it is immediate to see that the Dynkin diagram of type A_n is nondegenerate whenever n is even.

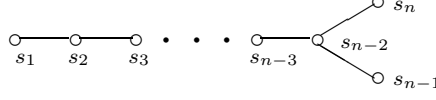(ii) Suppose that Γ is of type D_n as follows.

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Figure 6: the Dynkin diagram of type D_n.

Note that the two connected components of $\Gamma - \{s_{n-2}, s_{n-1}\}$ are of types A_1 and A_{n-3}. Since the Dynkin diagram of type A_1 is degenerate and by Proposition 3.2, the Dynkin diagram of type D_n is degenerate for all n.

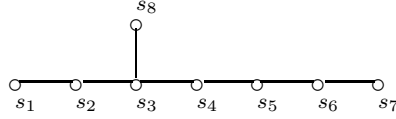(iii) Suppose that Γ is of type E_8 as follows.

Figure 7: the Dynkin diagram of type E_8.

Note that the connected graph $\Gamma - \{s_6, s_7\}$ is of type E_6. Since the Dynkin diagram of type E_6 is nondegenerate and by Proposition 3.2, it is immediate to see that the Dynkin diagram of type E_8 is nondegenerate.

The following corollary is useful for the main theorem. Note that the only nondegenerate trees of order at most four are the Dynkin diagrams of types A_2 and A_4.

Corollary 3.4. *Suppose that $\Gamma = (S, R)$ is a nondegenerate tree of order at least four. Then Γ has at least two such edges $tt' \in R$, where t is a leaf and t' has degree two.*

Proof. Proceed by induction on the order n of Γ. If $n = 4$, then the only nondegenerate tree is of type A_4. Hence the assertion holds if $n = 4$. Suppose that Γ is a nondegenerate tree of order $n > 4$. Pick an edge $ss' \in R$ with s a leaf. By induction hypothesis and Proposition 3.2, each connected component of $\Gamma - \{s, s'\}$ has at least two such edges or is of type A_2. This ensures that Γ has at least two such edges: if the degree of s' is two, then the connected graph $\Gamma - \{s, s'\}$ contains another; if not, every connected component of $\Gamma - \{s, s'\}$ contains one. Thus, this assertion holds by mathematical induction. $\qquad\square$

Recall that the form B corresponds to the linear transformation θ_B of V into V^*, which assigns to every $\lambda \in V$ the linear form $\mu \mapsto B(\lambda, \mu)$ on

V. Note that the kernel of θ_B is the radical $V^\perp$ of B. Hence the graph Γ is nondegenerate if and only if the linear transformation θ_B is invertible.

Recall that a *perfect matching* in a simple graph $\Gamma = (S, R)$ is a subset of the edge set R such that every vertex of Γ has exactly one edge incident on it. The linear transformation $\theta_B : V \to V^*$ with respect to the bases $\{\alpha_s \mid s \in S\}$ of V and $\{f_s \mid s \in S\}$ of V^* is equal to the adjacency matrix of the graph Γ over $\mathbf{F}_2$, since

$$\theta_B(\alpha_s) = \sum_{ss' \in R} f_{s'} \tag{3.1}$$

for $s \in S$. From the knowledge of algebraic graph theory, it is immediate to see the graph-theoretic condition of a nondegenerate tree as a tree with a perfect matching. See [3, Section 8.2] or [5, Section 2.1] for details. Through Proposition 3.2, there is a combinatorial proof of this. The paper omits the details because the proof of the main theorem doesn't use this.

4 The combinatorial representation of W of the second kind

This section shows that the lit-only σ-game on Γ corresponds to the dual representation ϕ of the combinatorial representation φ of W of the first kind. Let V^* denote the dual space of V, and let $\{f_s \in V^* \mid s \in S\}$ denote the dual basis of $\{\alpha_s \mid s \in S\}$. More precisely, $f_s(\alpha_s) = 1$ and $f_s(\alpha_{s'}) = 0$ if $s' \neq s$. For an endomorphism σ of V, the *transpose* ${}^t\sigma$ of σ is an endomorphism of V^* defined by ${}^t\sigma(f) = f\sigma$ for $f \in V^*$. Recall that the *dual representation* $\phi : W \to \mathrm{GL}(V^*)$ of the combinatorial representation φ of W of the first kind is defined by

$$\phi(w) := {}^t\varphi(w^{-1})$$

for $w \in W$. The representation ϕ is called the *combinatorial representation of W of the second kind* in this paper. Recall that V^* is a W-module by defining

$$wf := \phi(w)f$$

for $w \in W$ and $f \in V^*$. The following proposition gives a formula for the generators $\phi(s)$ of $\phi(W)$ for all $s \in S$.

Proposition 4.1. *For $s \in S$, the endomorphism $\phi(s)$ of V^* satisfies that*

$$\phi(s)f = f - f(\alpha_s) \sum_{ss' \in R} f_{s'} \tag{4.1}$$

for $f \in V^*$.

Proof. Pick any $s \in S$, $f \in V^*$ and $\lambda \in V$. A direct calculation shows that

$$\begin{aligned}
(\phi(s)f)(\lambda) &= (f\varphi(s))(\lambda) \\
&= f(\varphi(s)\lambda) \\
&= f(\lambda - B(\alpha_s, \lambda)\alpha_s) \\
&= f(\lambda) - f(\alpha_s)B(\alpha_s, \lambda) \\
&= (f - f(\alpha_s)\theta_B(\alpha_s))(\lambda).
\end{aligned}$$

Hence $\phi(s)f = f - f(\alpha_s)\theta_B(\alpha_s)$. By (3.1), we obtain the formula (4.1). $\qquad\square$

Each $f = \sum_{s \in S} c_s f_s$ of V^* is regarded as the configuration of lit-only σ-game on Γ consisting of all black vertices $s \in S$ with $c_s = 1$. From the formula (4.1), we see that the natural action of $\phi(W)$ on V^* coincides with the lit-only σ-game on Γ. As a result, it is reasonable that the lit-only σ-game on Γ is regarded as the *dual game* of the Reeder's game on Γ, and vice versa. Note that the subgroup $\phi(W)$ of $\mathrm{GL}(V^*)$ is called the *flipping group* of Γ in the paper [10].

Recall that the form B is W-invariant; i.e., $B(w\lambda, w\mu) = B(\lambda, \mu)$ for $w \in W$ and $\lambda, \mu \in V$. The following proposition shows an important property of the corresponding linear transformation θ_B of B.

Proposition 4.2. *The linear transformation $\theta_B : V \to V^*$ is a W-module homomorphism; i.e., $\phi(w)\theta_B = \theta_B\varphi(w)$ for all $w \in W$ as Figure 4.*

Figure 4: $\phi(w)\theta_B = \theta_B\varphi(w)$.

In particular, if Γ is nondegenerate then $\theta_B : V \to V^*$ is a W-module isomorphism.

Proof. Pick any $w \in W$ and $\lambda, \mu \in V$. A direct calculation shows that

$$\begin{aligned}
(w \cdot \theta_B(\lambda))(\mu) &= (\theta_B(\lambda)w^{-1})(\mu) \\
&= B(\lambda, w^{-1}\mu) \\
&= B(w\lambda, \mu) \\
&= \theta_B(w\lambda)(\mu)
\end{aligned}$$

Hence $\phi(w)\theta_B = \theta_B\varphi(w)$ for all $w \in W$. This completes the proof. $\qquad\square$

In the following section, the reader will see the use of Proposition 4.2 to lit-only σ-games in the case of nondegenerate B. Note that if the form B is degenerate, then there are more combinatorial properties of Reeder's game different from the lit-only σ-game. For example, the configuration consisting of all white vertices is the unique fixed configuration of the lit-only σ-game on Γ. However, each vector in $V^\perp$ is corresponding to a fixed configuration of the Reeder's game on Γ. In other words, the two representations ϕ and φ are isomorphic if and only if the graph Γ is nondegenerate.

5 Lit-only σ-game on a nondegenerate tree

The purpose of this section is to show the main result of this paper. Recall that a *W-orbit* of V (resp. V^*) is the subset $W\lambda := \{w\lambda \mid w \in W\}$ of V (resp. $Wf := \{wf \mid w \in W\}$ of V^*) for some $\lambda \in V$ (resp. $f \in V^*$). A W-orbit of V (resp. V^*) is said to be *trivial* if it consists of exactly one vector of V (resp. V^*). Otherwise, a W-orbit of V (resp. V^*) is said to be *nontrivial*. For example, $\{\lambda\}$ is a trivial W-orbit of V for $\lambda \in V^\perp$, and $\{0\}$ is the unique trivial W-orbit of V^*. This section begins with two lemmas.

Lemma 5.1. *The map which assigns to every W-orbit $W\lambda$ of V the subset $\theta_B(W\lambda)$ of V^* is between the W-orbits of V and the W-orbits of V^*. In particular, if Γ is nondegenerate, then the map is a bijection.*

Proof. Pick any $\lambda \in V$. By Proposition 4.2, we see $\theta_B(w\lambda) = w(\theta_B(\lambda))$ for all $w \in W$. Hence $\theta_B(W\lambda) = W(\theta_B(\lambda))$ is a W-orbit of V^*. This completes the proof. $\qquad\square$

Lemma 5.2. *([14, p.32]) Let the quadratic form $Q : V \to \mathbf{F}_2$ be defined by*

$$Q(\sum_{s \in S} c_s \alpha_s) := \sum_{s \in S} c_s^2 + \sum_{ss' \in R} c_s c_{s'}, \tag{5.1}$$

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where $c_s \in \mathbf{F}_2$ for $s \in S$ and the first and second sums are taken over all vertices and edges of Γ respectively. Then the following (i), (ii) hold.

(i) $Q(\lambda + \mu) = Q(\lambda) + Q(\mu) + B(\lambda, \mu)$ for $\lambda, \mu \in V$.

(ii) $Q(w\lambda) = Q(\lambda)$ for $w \in W$ and $\lambda \in V$. □

Lemma 5.2(ii) implies that $Q(\lambda) = Q(\mu)$ if λ and μ are in the same W-orbit of V. From this, the paper provides an alternative proof of the following lemma due to J. Goldwasser, X. Wang and Y. Wu.

Lemma 5.3. *([7, Theorem 6]) Suppose that Γ is a nondegenerate graph. For $f \in V^*$ and $s \in S$, if $f(\alpha_s) = 0$ then f and $f + \theta_B(\alpha_s)$ lies in distinct W-orbits of V^*.*

Proof. Note that $Q(\alpha_s) = 1$ for $s \in S$ by (5.1). A direct calculation shows that

$$
\begin{aligned}
Q(\theta_B^{-1}(f + \theta_B(\alpha_s))) &= Q(\theta_B^{-1}(f) + \alpha_s) \\
&= Q(\theta_B^{-1}(f)) + Q(\alpha_s) + B(\theta_B^{-1}(f), \alpha_s) \\
&= Q(\theta_B^{-1}(f)) + 1 + f(\alpha_s) \\
&= Q(\theta_B^{-1}(f)) + 1.
\end{aligned}
$$

by Lemma 5.2(i). Hence $\theta_B^{-1}(f + \theta_B(\alpha_s))$ and $\theta_B^{-1}(f)$ lie in distinct W-orbits of V. By Lemma 5.1, we see f and $f + \theta_B(\alpha_s)$ in distinct W-orbits of V^*. □

The following lemma, a consequence of [14, Lemma 2.2] and [14, Theorem 7.3], plays an important role in the main theorem of this paper.

Lemma 5.4. *([14]) Suppose that Γ is a nondegenerate tree and not a Dynkin diagram of type A_n for some even n. Then there are exactly three W-orbits of V : $\{0\}$, $\{\lambda \in V \mid Q(\lambda) = 0\}$ and $\{\lambda \in V \mid Q(\lambda) = 1\}$.* □

We now can prove the main theorem. Note that Examples 3.1 and 3.3 show that the nondegenerate trees include types A_n for even n, E_6 and E_8 among all simply-laced Dynkin diagrams. Hence the main theorem can be regarded as an extension of the lit-only σ-game on E_6 or E_8.

Theorem 5.5. *Suppose that $\Gamma = (S, R)$ is a nondegenerate tree and not a Dynkin diagram of type A_n for some even n. Then there are exactly three W-orbits of V^* :*

$$\{0\}, \ \{\theta_B(\lambda) \mid \lambda \in V \text{ and } Q(\lambda) = 0\}, \ \{\theta_B(\lambda) \mid \lambda \in V \text{ and } Q(\lambda) = 1\}.$$

Moreover, nontrivial W-orbits of V^* both contain f_s for some $s \in S$.

Proof. It is immediate from Lemmas 5.1 and 5.4 that the W-orbits of V^* are $\{0\}$, $\{\theta_B(\lambda) \mid \lambda \in V \text{ and } Q(\lambda) = 0\}$ and $\{\theta_B(\lambda) \mid \lambda \in V \text{ and } Q(\lambda) = 1\}$. It remains to show that $\{\theta_B(\lambda) \mid \lambda \in V \text{ and } Q(\lambda) = 0\}$ and $\{\theta_B(\lambda) \mid \lambda \in V \text{ and } Q(\lambda) = 1\}$ both contain f_s for some $s \in S$. Recall that the Dynkin diagrams of types A_2 and A_4 are all nondegenerate trees of order at most four. Hence Γ has order $n > 4$. By Corollary 3.4, there is such an edge $tt' \in R$, where t is a leaf and t' has degree two. Let s be the other neighbor of t'. By Lemma 5.3, we see that $f_s = f_t + \theta_B(\alpha_{t'})$ lies in one nontrivial W-orbit of V^* and f_t lies in the other. This completes the proof. $\square$

In terms of combinatorics, Theorem 5.5 says that if Γ is a nondegenerate tree and not of type A_n for some even n, then for an initial configuration of lit-only σ-game on Γ with at least one black vertex, one can find a sequence of moves to reach a configuration with a single black vertex. The paper [11] shall give a generalization of Theorem 5.5: if Γ is a nondegenerate graph and not a line graph, then there are exactly three W-orbits of V^*, and nontrivial W-orbits of V^* both contain f_s or $f_s + f_{s'}$ for some distinct $s, s' \in S$.

6 Problems

The purpose of this section is to provide some open problems and to review the progress in lit-only σ-games from the views of algebra and combinatorics.

Problems 6.1. *Classify all graphs Γ with the combinatorial property: each nontrivial W-orbit of V^* contains f_s for some $s \in S$. Note that if Γ is a line graph, the description of W-orbits of V^* can be found in [8, 18]. The problem may firstly consider the case of line graphs.*

Problems 6.2. *Give a presentation of $\phi(W)$ using these generators $\phi(s)$ for all $s \in S$. The paper [10] gives a little advancement for the problem, which determines that the kernel ϕ is the center of the Coxter group W in the case of simply-laced Dynkin diagrams.*

Problems 6.3. *Classify the flipping groups $\phi(W)$ of Γ up to isomorphism. Recall that the quadratic form Q is said to be nonsingular if $Q(\lambda) = 1$ for all $\lambda \in V^\perp - \{0\}$. The paper [14] defines that the graph Γ is said to be nonsingular if Q is nonsingular. [14, Theorem 7.3] shows that if Γ is a nonsingular tree not of type A_n, then $\varphi(W) \simeq \phi(W)$ is the orthogonal group $\{a \in \mathrm{GL}(V) \mid Q(a\lambda) = Q(\lambda) \text{ for all } \lambda \in V\}$. The paper [18] shows that the*

flipping group of the line graph of a tree of order $n \geq 3$ is isomorphic to the symmetric group of degree n. Afterward, the paper [8] knows that the structure of the flipping group of any line graph. The paper [9] shows that in some class of 2^{n-1} graphs, the number of flipping groups is at most $n-1$ up to isomorphism.

Problems 6.4. *Give a combinatorial proof for Theorem 5.5. Note that if Γ is a nondegenerate tree, then the main result of [17] implies that each nontrivial W-orbit of V^* contains f_s or $f_s + f_{s'}$ for some distinct $s, s' \in S$.*

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